SIAM J. MATH. ANAL.
Vol. 17, No. 4, July 1986

007

ON THE EIGENVALUE PROBLEM FOR COUPLED ELLIPTIC SYSTEMS*

ROBERT STEPHEN CANTRELL† AND KLAUS SCHMITT‡

Abstract. We consider the eigenvalue problem

$$L_k u_k = \lambda \sum_{i=1}^{r} m_{ki} u_i \quad \text{in } \Omega,$$
$$u_k = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqq 1$, with smooth boundary $\partial\Omega$ and for $k = 1, \cdots, r$, L_k is a second order uniformly elliptic operator. The coupling coefficients are such that $m_{ij} \geqq 0$, $i \neq j$ and for at least one k, $m_{kk}^{+} \neq 0$. We establish the existence of positive characteristic values with associated positive solutions. We also investigate the multiplicity of such characteristic values and establish bifurcation results for nonlinear perturbations of the linear problem.

Key words. coupled elliptic systems, eigenvalue problems, bifurcation in nonlinear systems

AMS(MOS) subject classifications. Primary 35J55, 35F30, 35J65

1. Introduction.

Consider the eigenvalue problem

$$(1.1) \qquad \begin{aligned} L_k u_k &= \lambda \sum_{i=1}^{r} m_{ki} u_i \quad &&\text{in } \Omega, \\ u_k &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geqq 1$, with smooth boundary $\partial\Omega$, and for $k = 1, \cdots, r$

$$(1.2) \qquad L_k = \sum_{i,j=1}^{n} a_{ij}^{k} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i^k \frac{\partial}{\partial x_i} + a_0^k$$

is a uniformly elliptic differential operator of second order with coefficients continuous on $\bar{\Omega}$ and $a_0^k(x) \geqq 0$, $x \in \bar{\Omega}$. The coefficients m_{ki}, $1 \leqq k$, $i \leqq r$ are also assumed to belong to $C^0(\bar{\Omega}, \mathbb{R})$. The parameter $\lambda \in \mathbb{R}$ is assumed to be positive.

In a recent paper, P. Hess [11] showed that if $m_{ij} \geqq 0$, $i \neq j$ and if for at least one k, $m_{kk}^{+} \neq 0$, then (1.1) has a positive characteristic value with associated nontrivial solution $u = \text{col}(u_1, \cdots, u_r) \in K = \{v \in C^0(\bar{\Omega}, \mathbb{R}^r): v_i \geqq 0, 1 \leqq i \leqq r\}$. The purpose of this paper is to examine this important result more closely. We obtain a somewhat more detailed understanding of the multiplicity and character of the nontrivial solutions to (1.1), leading to results on bifurcation questions for associated nonlinear eigenvalue problems.

* Received by the editors April 24, 1984, and in revised form February 5, 1985. This research was supported in part by the National Science Foundation under grant MCS 8121951.
† Department of Mathematics and Computer Science, University of Miami, Coral Gables, Florida 33124.
‡ Department of Mathematics, University of Utah, Salt Lake City, Utah 84112.

To this end, we shall begin by proving the main result of Hess [11] in a slightly different way, relying on ideas employed earlier by the second author in [18]. We then give an extension of the result to the multiparameter problem

$$(1.3)\qquad \begin{aligned} &L_k u_k = \lambda_k \sum_{i=1}^{r} m_{ki} u_i && \text{in } \Omega, \\ &u_k = 0 && \text{on } \partial\Omega, \quad k = 1, \cdots, r. \end{aligned}$$

Together with some conditions for the uniqueness and simplicity of such characteristic values of (1.1), our result on (1.3) explains how multiplicities greater than one occur in a number of cases. Finally, we apply these results to the problem of positive solutions to nonlinear eigenvalue problems, including the problem of coexistence steady states in the Volterra–Lotka competition model with diffusion, recently studied by Cosner and Lazer [8].

2. Main results. Let L_k, $1 \leqq k \leqq r$ also denote the realization of L_k in $C_0^0(\overline{\Omega}, \mathbb{R})$ subject to Dirichlet boundary conditions. Then L_k: $C_0^0(\overline{\Omega}, \mathbb{R}) \supset \operatorname{dom}(L_k) \to C_0^0(\overline{\Omega}, \mathbb{R})$ is invertible, with compact inverse. Furthermore, L_k^{-1} is a positive operator with respect to the cone of nonnegative functions. Denote by M the matrix $M = (m_{ij})$, $1 \leqq i$, $j \leqq r$ and think of M as a multiplication operator. (Recall that $m_{ij} \geqq 0$ if $i \neq j$.) Then (1.1) may be written as

$$(2.1)\qquad Lu = \lambda Mu,$$

where

$$L = \begin{bmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_r \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix},$$

and L^{-1} is a compact operator on $C_0^0(\overline{\Omega}, \mathbb{R}^r)$ which is positive with respect to the cone K in $C_0^0(\overline{\Omega}, \mathbb{R}^r)$, i.e. $L^{-1}(K) \subset K$.

Let us choose $\mu > 0$ such that all elements on the main diagonal of $M + \mu I = M + \mu$, where I is the $r \times r$ identity matix, are positive. Then (2.1) is equivalent to

$$(2.2)\qquad (L + \lambda\mu) u = \lambda (M + \mu) u,$$

and since μ and λ are positive $(L + \mu\lambda)^{-1}$ is also a compact operator positive with respect to K. Thus (2.2) is equivalent to

$$(2.3)\qquad u = \lambda (L + \lambda\mu)^{-1} (M + \mu) u.$$

We let $A_\lambda = \lambda(L + \lambda\mu)^{-1}(M + \mu)$. The following result then holds.

LEMMA 2.1. *Let* $r(A_\lambda)$ *denote the spectral radius of* A_λ. *Then the mapping* $\lambda \to r(A_\lambda)$ *is continuous on* $(0, \infty)$ *with* $\lim_{\lambda \to 0^+} r(A_\lambda) = 0$.

Proof. The map A_λ depends continuously on λ in the strong operator topology. Since the family $\{A_\lambda\}$ is a compact family, it follows from a result of Nussbaum [13] that the map $\lambda \to r(A_\lambda)$ is continuous.

LEMMA 2.2. *Assume* $m_{kk}^+ \neq 0$ *for some* $k \in \{1, \cdots, r\}$. *Then there exists* $\lambda > 0$ *such that* $r(A_\lambda) \geqq 1$.

Proof. According to Hess–Kato [12], there exists $\lambda > 0$ and $u_k \in C_0^0(\overline{\Omega}, \mathbb{R})$, $u_k(x) > 0$, $x \in \Omega$ such that

$$L_k u_k = \lambda m_{kk} u_k.$$

Letting $u=\mathrm{col}(0,\cdots,u_k,0,\cdots,0)$ one gets $Lu\leqq\lambda Mu$. Thus $(L+\lambda\mu)u\leqq\lambda(M+\mu)u$. Hence for this value of λ

$$u\leqq\lambda(L+\lambda\mu)^{-1}(M+\mu)u,$$

i.e. $u\leqq A_\lambda u$.

Iterating this inequality, we get $u\leqq A_\lambda^n u$, and since the C_0^0 norm is monotone with respect to the cone K, we get

$$|u|\leqq|A_\lambda^n||u|.$$

Hence $1\leqq|A_\lambda^n|^{1/n}$. Thus $r(A_\lambda)\geqq 1$.

THEOREM 2.3. *Let* $m_{kk}^+\neq 0$ *for some* $k\in\{1,2,\cdots,r\}$. *Then there exists a smallest* $\bar\lambda>0$ *and* $u\in K\setminus\{0\}$ *such that*

$$u=A_{\bar\lambda}u,$$

i.e. $Lu=\bar\lambda Mu$.

Proof. Since $r(A_\lambda)$ is continuous and $r(A_\lambda)\to 0$ as $\lambda\to 0$ and since by Lemma 2.2, there exists λ such that $r(A_\lambda)\geqq 1$, it follows that there exists a smallest $\bar\lambda$ such that

$$r(A_{\bar\lambda})=1.$$

Since $A_{\bar\lambda}$ is positive and compact and K is total one may employ the theorem of Krein–Rutman (see [2]) to conclude that there exists $u\in K\setminus\{0\}$ such that

$$u=r(A_{\bar\lambda})u=A_{\bar\lambda}u,$$

i.e., $Lu=\bar\lambda Mu$.

COROLLARY 2.4. *If* $\lambda>0$ *is any other characteristic value, then* $\lambda\geqq\bar\lambda$.

Proof. Let λ be a characteristic value. Then there is $u\neq 0$ such that $u=A_\lambda u$. Iterating, one obtains $u=A_\lambda^n u$. Hence $1\leqq|A_\lambda^n|^{1/n}$, implying that $r(A_\lambda)\geqq 1$. The result then follows from the proof of Theorem 2.3.

Now consider (1.3). Assume that $(\lambda_1,\cdots,\lambda_r)$ is restricted to a ray emanating from the origin of $\mathbb{R}^r$ into the positive cone. Theorem 2.3 then obtains in most cases. To see this, observe that if $(\lambda_1,\cdots,\lambda_r)$ is as restricted, (1.3) is equivalent to

$$Lu=t\tilde{M}u, \tag{2.4}$$

where

$$\tilde{M}=\begin{bmatrix}\lambda_1^0 & & \\ & \ddots & \\ & & \lambda_r^0\end{bmatrix}M,\qquad t\in I,$$

$\lambda_1^0\geqq 0$ and $(\lambda_1^0)^2+\cdots+(\lambda_r^0)=1$. The result follows provided $\tilde{m}_{kk}^+=\lambda_k^0 m_{kk}^+\neq 0$ for some $k\in\{1,2,\cdots,r\}$.

Suppose now that $\lambda=(\lambda_1,\cdots,\lambda_r)$ is such that $\lambda_i\geqq 0$ and $|\lambda|^2>0$. Define

$$A_\lambda=|\lambda|^2\left(L+|\lambda|^2\mu\right)^{-1}(\tilde{M}+\mu)$$

with

$$\tilde{M}=\frac{1}{|\lambda|^2}\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_r\end{bmatrix}M.$$

If $|\lambda^0|^2=1$, and $\lambda_i^0>0$, $i=1,\cdots,r$, we define $F(\lambda^0)=\bar{t}(\lambda^0)$, where $\bar{t}>0$ is the smallest number such that $r(A_{\bar{t}\lambda^0})=1$. There are a number of conditions under which the function F is continuous. In particular, we have the following result.

THEOREM 2.5. *F is continuous at $\lambda^0\in(\mathbb{R}_+)^r\cap S^{r-1}$ provided any one of the following conditions holds*:

i) $m_{ij}\geqq 0$ *for* $i,j=1,\cdots,k$.

ii) *If* $F(\lambda^0)=\bar{t}\lambda^0$, *then* $\bar{t}$ *is the only positive number* t *such that* $u=A_{t\lambda^0}u$ *has a nontrivial solution* $u\in K$.

Proof. In case (i), $r(A_{t\lambda^0})$ is a nondecreasing function of t, $t\geqq 0$. The result follows from an application of a result of Nussbaum [13]. In case (ii), one uses a simple compactness argument.

3. Coupling and multiplicity. In this section we investigate the question of uniqueness and multiplicity of the characteristic values of (1.1). We begin with the following lemma.

LEMMA 3.1. *Suppose $\mu\geqq 0$ is chosen so that all entries of the matrix $M+\mu$ are nonnegative and that there exists $x_0\in\Omega$ such that $(M+\mu)(x_0)$ is irreducible. Then if $Lu=\lambda Mu$ and $u\in K\backslash\{0\}$, then in fact $u\in\operatorname{int}K$ (where* int *is with respect to the $C^1(\bar{\Omega})$ topology).*

Proof. Since $Lu=\lambda Mu$ we have that

$$(L+\lambda\mu)u=\lambda(M+\mu)u,$$

and componentwise

$$(L_i+\lambda\mu)u_i=\lambda\sum_{j=1}^{r}(m_{ij}+\mu\delta_{ij})u_j\geqq 0.$$

It then follows from the Hopf maximum principle that for each $i\in\{1,\cdots,r\}$, either $u_i\equiv 0$ on $\bar{\Omega}$ or $u_i(x)>0$, for all $x\in\Omega$.

Since $u\in K\backslash\{0\}$, there must be at least one $i_0\in\{1,\cdots,r\}$ such that $u_{i_0}(x)>0$ on Ω. If there is no other such i, $(M+\mu)(x_0)$ has a one-dimensional invariant subspace, contradicting the assumption of irreducibility. Hence there is $i_1\in\{1,\cdots,r\}$, $i_1\neq i_0$ such that $u_{i_1}(x)>0$ on Ω. If i_0 and i_1 are the only such, then $(M+\mu)(x_0)$ has a two-dimensional invariant subspace, again a contradiction. Iterating this argument now guarantees $u_i(x)>0$ on Ω for $i=1,\cdots,r$. It further follows from [10, Lemma 3.4] that $\partial u_i/\partial\nu$ is negative on $\partial\Omega$ for each i, proving the result.

THEOREM 3.2. *Suppose there are $\mu\geqq 0$ and $x_0\in\Omega$ such that $(M+\mu)(x_0)$ is irreducible. Then if $\bar{\lambda}$ is as in Theorem 2.3, $\bar{\lambda}$ is a geometrically simple eigenvalue.*

Proof. Suppose u and v are nontrivial solutions of (1.1) with $u\in K$. Lemma 3.1 implies $u\in\operatorname{int}K$. Hence for small δ, $u-\delta v\in\operatorname{int}K$. Let $\delta^*=\sup\{\delta>0\colon u-\delta v\in\operatorname{int}K\}$, $\delta_*=\inf\{\delta<0\colon u-\delta v\in\operatorname{int}K\}$. Since $v\neq 0$ at least one of δ^* or δ_* must be finite. With no loss of generality, assume $\delta^*<\infty$. Then $u-\delta^*v\in\partial K$. Lemma 3.1 implies $u\equiv\delta^*v$.

LEMMA 3.3. *Suppose there are $\mu\geqq 0$ and $x_0\in\Omega$ such that $(M+\mu)(x_0)$ is irreducible. Then if $\bar{\lambda}$ is as in Theorem 2.3, $(I-A_{\bar{\lambda}})^2z=0$ implies $(I-A_{\lambda})z=0$.*

Proof. Suppose $(I-A_{\bar{\lambda}})^2z=0$. Then Theorem 3.2 implies that $(I-A_{\bar{\lambda}})z=cu$, where $u=A_{\bar{\lambda}}u$. Let $A_{\bar{\lambda}}^*$ denote the Banach space adjoint of $A_{\bar{\lambda}}$ considered in $C_0(\bar{\Omega},\mathbb{R}^r)$. The Krein–Rutman theorem implies there is a continuous linear functional f^* (with $f^*(K)\subset[0,\infty]$ and $f^*(\operatorname{int}K)\subset(0,\infty)$) such that $A_{\bar{\lambda}}^*f^*=f^*$. Hence

$$f^*z-f^*A_{\bar{\lambda}}z=cf^*(u)$$

which implies $0=cf^*(u)$. Hence, since $u\in\operatorname{int}K$, $c=0$.

THEOREM 3.4. *Suppose $m_{ii} \geqq 0$ for $i=1,\cdots,r$. Then if $M(x_0)$ is irreducible for some $x_0 \in \Omega$ and $\bar{\lambda}$ is as in Theorem 2.3, $\bar{\lambda}$ is an algebraically simple eigenvalue and the only positive eigenvalue admitting a solution u, where $u \in K$.*

Proof. That $\bar{\lambda}$ is an algebraically simple eigenvalue is a consequence of Lemma 3.3. Suppose now $\lambda' > \bar{\lambda}$ is such that $v = \lambda' L^{-1} M v$, with $v \in K$. Then Lemma 3.1 implies $v \in \operatorname{int} K$. Furthermore, since $r(A_\lambda)$ is a strictly increasing function of λ, $r(A_{\lambda'}) > 1$. The Krein–Rutman theorem implies the existence of $\tilde{v} \in K \setminus \{0\}$ such that $r(A_{\lambda'})\tilde{v} = A_{\lambda'}\tilde{v}$. Applying Lemma 3.1 to the equation

$$L\tilde{v} = \frac{\lambda'}{r(A_{\lambda'})} M\tilde{v},$$

we conclude that in fact $\tilde{v} \in \operatorname{int} K$. For $\delta > 0$ and sufficiently small, $v - \delta\tilde{v} \in \operatorname{int} K$. Let $\delta^* = \sup\{\delta > 0\colon\ v - \delta\tilde{v} \in \operatorname{int} K\}$. Since $\tilde{v} \in \operatorname{int} K$, $\delta^* < \infty$ and $v - \delta^*\tilde{v} \in K$, i.e., $\delta^*\tilde{v} \leqq v$. Hence $\delta^* r(A_{\lambda'})\tilde{v} = \delta^* A_{\lambda'} v \leqq A_{\lambda'} v = v$. It follows that $\delta^* r(A_{\lambda'}) \leqq \delta^*$, and so $r(A_{\lambda'}) \leqq 1$, a contradiction.

We do not know in general whether Theorem 3.4 remains valid if the assumption $m_{ii} \geqq 0$ for $i=1,\cdots,r$ is removed. However, with some additional restrictions on the system (1.1), the theorem remains valid. As we shall see, the restrictions are substantial. Nevertheless, the result is quite useful from the point of view of applications to nonlinear analysis. Before stating the result, we give two lemmas which will be needed in the proof.

LEMMA 3.5. *Suppose that $m_{ij}(x) \leqq 1/r$, $i \neq j$, and that $-1 \leqq m_{ii} \leqq -1 + 1/r$ for $j, i = 1,\cdots,r$. Then there is no positive eigenvalue for* (1.1) *admitting a solution in $K \setminus \{0\}$.*

Proof. The result follows from an application of the maximum principle. See [14, pp. 188–192].

LEMMA 3.6. *Let $m_{ii}+1>0$ for $i=1,\cdots,r$, let $M+I$ be irreducible, and assume there is $\lambda > 0$ and $u \in K\setminus\{0\}$ such that $Lu = \lambda Mu$. Then $N((I-\lambda L^{-1}M)^2) = N(I-\lambda L^{-1}M) = \langle u \rangle$, whenever $L^{-1}M = ML^{-1}$.*

Proof. That $N(I-\lambda L^{-1}M) = \langle u \rangle$ follows from the fact that $u \in \operatorname{int} K$ (see Lemma 3.1). Consider $\lambda(L+\lambda)^{-1}(M+1)$. By the proof of Theorem 3.4, $r(\lambda(L+\lambda)^{-1}(M+1)) = 1$. If A_λ^* denotes the Banach space adjoint of $\lambda(L+\lambda)^{-1}(M+1)$, the Krein–Rutman theorem guarantees the existence of continuous linear functional f^* such that $A_\lambda^* f^* = f^*$ and such that $f^*(\operatorname{int} K) \subset (0,\infty)$.

Suppose $(L-\lambda M)^2 x = 0$. Then $Lx - \lambda Mx = cu$, for some $c \in \mathbb{R}$. Hence $(L+\lambda)x - \lambda(M+1)x = cu$, or equivalently, $x - \lambda(L+\lambda)^{-1}(M+1)x = c(L+\lambda)^{-1}u$. It follows that $f^*x - f^*(\lambda(L+\lambda)^{-1}(M+1)x) = cf^*((L+\lambda)^{-1}u)$. Now $f^*(\lambda(L+\lambda)^{-1}(M+1)x) = (A_\lambda^* f^*)x = f^*x$. So $0 = cf^*((L+\lambda)^{-1}u)$. Since $u \in \operatorname{int} K$, $c=0$.

Finally, if $L^{-1}M = ML^{-1}$ and $(I-\lambda L^{-1}M)^2 x = 0$, then a simple computation shows $(L-\lambda M)^2 x = 0$. Hence $(L-\lambda M)x = 0$, or equivalently, $(I-\lambda L^{-1}M)x = 0$.

Remark 3.7. We note that in Lemma 3.6 that we do not need $m_{ii} > 0$ for some $i \in \{1,2,\cdots,r\}$.

THEOREM 3.8. *Suppose that the conditions of Theorem 2.3 are satisfied. In addition, assume*

(i) *$m_{ij} \leqq 1/r$, if $i \neq j$;*

(ii) *$-1/2r < m_{ii} < 1/2r$, for $i=1,\cdots,r$.*

If $L^{-1}M = ML^{-1}$, $(M+I)$ is irreducible, and if $\bar{\lambda}$ is as in Theorem 2.3, then $\bar{\lambda}$ is a simple eigenvalue for (1.1) *and the only positive eigenvalue admitting a solution u, with $u \in K\setminus\{0\}$.*

Remark 3.9. (a) Since (1.1) may be rescaled, conditions (i) and (ii) above are restrictions only on the relative sizes of the diagonal versus off-diagonal terms of the matrix M. The commutativity condition requires that $L_i = L_j$ for $i,j = 1, \cdots, r$, and that M be constant, although M may have negative entries on its main diagonal.

(b) The proof relies on an "unfolding" of the problem in a manner analogous to that employed in [12]. We also obtain partial results in case $(M+I)(x_0)$ is irreducible for some $x_0 \in \Omega$ (dropping the commutativity assumption).

Proof of Theorem 3.8. Assume initially only that M satisfies conditions (i) and (ii) of the hypotheses and that $(M+I)(x_0)$ is irreducible for some $x_0 \in \Omega$. Let $\lambda > 0$ and $t \in \mathbb{R}$ and define $A_{\lambda,t}$ by

$$A_{\lambda,t} = \lambda(L+\lambda)^{-1}(M-t+1).$$

We first observe that there is $t^* \in (0, 1-1/2r)$ such that $A_{\lambda,t}$ is a positive operator for $\lambda > 0$ and $t \leq t^*$ and that the equation

$$u = A_{\lambda,t}u \tag{3.1}$$

has no solution with $\lambda > 0$, $t = t^*$, and $u \in K \setminus \{0\}$. To see that this is the case, consider

$$-\frac{1}{2r} < \min_{x\in\bar{\Omega}} m_{ii}(x) \leq \max_{x\in\bar{\Omega}} m_{ii}(x) < \frac{1}{2r}$$

for some $i \in \{1, 2, \cdots, r\}$. It follows that if $0 < t < 1-1/2r$, then $\min_{x\in\bar{\Omega}} m_{ii}(x) - t > -1$, and that if $t > \max_{x\in\bar{\Omega}} m_{ii}(x) + 1 - \frac{1}{r}$, then $\max_{x\in\bar{\Omega}} -m_{ii}(x) - t \leq -1 + 1/r$. Since

$$\max_{x\in\bar{\Omega}} m_{ii}(x) + 1 - \frac{1}{r} < \frac{1}{2r} + 1 - \frac{1}{r} = 1 - \frac{1}{2r},$$

our observation follows from Lemma 3.5.

Next observe that if $t \leq 0$, Theorem 2.3 implies that there exists a smallest positive number $\bar{\lambda}(t)$ such that (3.1) has a solution $u \in K \setminus \{0\}$. If $t = t^*$, no such number exists. We now define a function f: $(-\infty, t^*] \to [0, \infty)$ by

$$f(t) = \begin{cases} 1/\bar{\lambda}(t) & \text{provided } \bar{\lambda}(t) \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose now $t < t' \leq t^*$ and there exists $\lambda(t') > 0$ and $u \in K \setminus \{0\}$ such that

$$u = \lambda(t')(L+\lambda(t'))^{-1}(M-t'+1)u.$$

Since $0 < m_{ii} - t' + 1 < m_{ii} - t + 1$ for $i = 1, \cdots, r$, we have

$$u \leq \lambda(t')(L+\lambda(t'))^{-1}(M-t+1)u.$$

It follows that $\bar{\lambda}(t)$ exists and $\bar{\lambda}(t) \leq \bar{\lambda}(t')$. Furthermore, if $f(t') = 0$, $f(t) = 0$ for $t \in [t', t^*]$. Hence f is a monotonic nonincreasing function and, as such, can have at most a countable number of discontinuities.

Suppose now that $\lambda_0 > 0$, $t_0 < t^*$, and $u \in K \setminus \{0\}$ such that

$$u = \lambda_0(L+\lambda_0)^{-1}(M-t_0+1)u.$$

Lemma 3.1 implies that $u \in \operatorname{int} K$. It follows from Lemma 3.3 that 1 is a simple eigenvalue of $\lambda_0(L+\lambda_0)^{-1}(M-t_0+1)$ and that $r(\lambda_0(L+\lambda_0)^{-1}(M-t_0-1)) = 1$. Since $r(\lambda(L+\lambda)^{-1}(M-t+1))$ depends continuously on λ and t, a perturbation theory argument will show that if $\alpha(\lambda,t)=r(\lambda(L+\lambda)^{-1}(M-t+1))$, α is an analytic function of (λ,t) in a neighborhood of (λ_0,t_0). Furthermore, one may choose an eigenfunction $u(\lambda,t)$ corresponding to $\alpha(\lambda,t)$ so that $u(\lambda,t)$ is analytic in (λ,t) also. (The perturbation theory argument necessary can be adapted from [17, pp. 57–64]. We note only that the simplicity of the eigenvalue 1 of $\lambda_0(L+\lambda_0)^{-1}(M-t_0+1)$ is essential to the argument.) Let us now consider the equation

$$\alpha u=\lambda(L+\lambda)^{-1}(M-t+1)u. \tag{3.2}$$

Differentiating (3.2) with respect to t and evaluating at (λ_0,t_0) we obtain

$$\begin{aligned}&\alpha_t(\lambda_0,t_0)u(\lambda_0,t_0)+u_t(\lambda_0,t_0)\\ &\quad=\lambda_0(L+\lambda_0)^{-1}(M-t_0+1)u_t(\lambda_0,t_0)-\lambda_0(L+\lambda_0)^{-1}u(\lambda_0,t_0).\end{aligned} \tag{3.3}$$

Let $A^*_{\lambda_0,t_0}$ be the Banach space adjoint of A_{λ_0,t_0}. The Krein–Rutman Theorem implies that there is a continuous linear functional f^* (with $f^*(\operatorname{int}K)\subset(0,\infty)$) such that $f^*A_{\lambda_0,t_0}=A^*_{\lambda_0,t_0}f^*=f^*$. Applying f^* to (3.3) yields

$$\alpha_t(\lambda_0,t_0)f^*(u(\lambda_0,t_0))=-\lambda_0 f^*\left[(L+\lambda_0)^{-1}u(\lambda_0,t_0)\right]. \tag{3.4}$$

Since $u(\lambda_0,t_0)\in\operatorname{int}K$, $\alpha_t(\lambda_0,t_0)\neq 0$. The Implicit Function Theorem implies the existence of $\delta>0$ and a smooth function g: $(\lambda_0-\delta,\lambda_0+\delta)\to\mathbb{R}$ such that $g(\lambda_0)=t_0$ and $\alpha(\lambda,g(\lambda))=1$.

Since f is nonincreasing, if $t_1<t_2$ and $f(t_1)=f(t_2)>0$, then $f(t)\equiv f(t_1)=1/\bar\lambda(t_1)$ for $t\in[t_1,t_2]$. So $\alpha(\bar\lambda(t_1),t)=1$ for $t\in[t_1,t_2]$ by Lemma 3.3. But for any $t_0\in(t_1,t_2)$, the preceding argument shows that the solution set to $\alpha(\lambda,t)=1$ is expressible as a function of λ in a neighborhood of $(\bar\lambda(t_1),t_0)$, a contradiction. Hence f is strictly decreasing so long as it remains positive.

Let $t^{**}\in(0,1-1/2r)$ be given by $t^{**}=\inf\{t\leqq t^*\colon f(t)=0\}$ and also let $\gamma=\lim_{t\to-\infty}f(t)$ and $0\leqq\omega=\inf\{f(t)\colon f(t)>0\}$. Since f is strictly decreasing, it has an inverse h defined from a subset of (ω,γ) into $(-\infty,t^{**})$. We claim that this function h is extendable to a continuous function $\tilde h$: $(0,\gamma)\to(-\infty,t^{**})$ such that if $s\in(0,\gamma)$, $\alpha(1/s,\tilde h(s))=1$.

Let us now establish this claim. Let $t_0<t^{**}$ and let

$$L_0=\lim_{t\to t_0^-}f(t)\geqq\lim_{t\to t_0^+}f(t)=R_0>0.$$

It follows from [13] that

$$\alpha\left(\frac{1}{L_0},t_0\right)=1\quad\text{and}\quad\alpha\left(\frac{1}{R_0},t_0\right)=1.$$

The minimality of $\bar\lambda(t_0)$ implies that $L_0=f(t_0)$. Notice that if $t\leqq-1/2r$, Theorem 3.4 implies that $L_0=R_0$. So if $L_0>R_0$, $t_0\in(-1/2r,t^{**})$. Furthermore, the Implicit Function Theorem may be applied as before at $(1/L_0,t_0)$ and $(1/R_0,t_0)$, giving functions g_1 and g_2 respectively. Notice that: if $\lambda\in[1/L_0,1/R_0]$ and $g_1(\lambda)$ is defined, then Theorem 3.4 and the minimality of $\bar\lambda(t)$ for $t>t_0$ implies that $g_1(\lambda)\in[-1/2r,t_0]$, and similarly for $g_2(\lambda)$. By [13], $g_1(\lambda)$ and $g_2(\lambda)$ can be extended on $[1/L_0,1/R_0]$. Since $A_{\bar\lambda,t}$ is monotonic in t for fixed $\bar\lambda$, so must $r(A_{\bar\lambda,t})$ be. Hence if $g_1(\lambda)\neq g_2(\lambda)$ for some $\lambda\in[1/L_0,1/R_0]$, $\alpha(\lambda,t)=1$ for t between $g_1(\lambda)$ and $g_2(\lambda)$, a contradiction to the Implicit Function Theorem. Hence $\tilde h$ may be defined on $[R_0,L_0]$ by $\tilde h(s)=g_1(1/s)$.

We have now shown the existence of $\tilde{h}$ on (ω,γ). If $\omega=0$, there is nothing more to do. If $\omega>0$, then $\alpha(1/\omega,t^{**})=1$ and the Implicit Function Theorem guarantees the existence of a g as before with $g(\lambda)\in[-1/2r,t^{**}]$ and $\alpha(\lambda,g(\lambda))=1$. Since $g(\lambda)\in[-1/2r,t^{**}]$, it again follows from the continuity of the spectral radius [13] that g is extendable to $(1/\omega,\infty)$. Defining $\tilde{h}(s)=g(1/s)$ for $s\in(0,\omega)$ completes the verification of the claim.

Now assume that $L^{-1}M=ML^{-1}$. Suppose there is a $t_0\in(-1/2r,t^{**})$ such that $L_0>R_0$. Now $1/L_0=\bar{\lambda}(t_0)<1/R_0<\bar{\lambda}(t)$ for $t>t_0$. Lemma 3.6 implies there exists a $\delta\in(0,1)$ such that the Leray–Schauder indices $\operatorname{ind}_{LS}(I-(1+\delta)\bar{\lambda}(t_0)L^{-1}(M-t_0))$ and $\operatorname{ind}_{LS}(I-(1-\delta)\bar{\lambda}(t_0)L^{-1}(M-t_0))$ are defined and unequal. We may also assume that $\delta>0$ is sufficiently small so that $((1+\delta)\bar{\lambda}(t_0)<1/R_0$. The homotopy invariance property of the Leray–Schauder degree guarantees that

$$\operatorname{ind}_{LS}\big(I-(1+\delta)\bar{\lambda}(t_0)L^{-1}(M-t)\big)\neq\operatorname{ind}_{LS}\big(I-(1-\delta)\bar{\lambda}(t_0)L^{-1}(M-t)\big)$$

for $t\in(t_0,t_0+\varepsilon)$ for $\varepsilon>0$ and sufficiently small. Hence for $t\in(t_0,t_0+\varepsilon)$ there exist $0<\lambda<\bar{\lambda}(t)$ with $N(I-\lambda L^{-1}(M-t))\neq\{0\}$, a contradiction. Hence $L_0=R_0$ and so f is continuous on $(-\infty,t^{**})$. A similar argument gives $\lim_{t\to t^{**}}f(t)=0$ in this case.

The uniqueness assertion of the theorem is now evident from the Implicit Function Theorem and the monotonicity of $r(A_{\lambda,t})$ in t.

COROLLARY 3.10. *Suppose M satisfies* (i) *and* (ii) *of Theorem* 3.8 *and* $(M+I)(x_0)$ *is irreducible for some $x_0\in\Omega$. Then if $\tilde{h}$, $A_{\lambda,t}$, and γ are as in the proof of Theorem* 3.8, *then the set* $\{(\lambda,t)\in(0,\infty)\times(-\infty,t^{**}]$: $u=A_{\lambda,t}u$ *for some* $u\in K\setminus\{0\}\}=\{(\lambda,h(\frac{1}{\lambda}))$: $\lambda\in(\frac{1}{\gamma},\infty)\}$.

The requirement that $(M+\mu)(x_0)$ be irreducible for some $x_0\in\Omega$ represents a rather strong coupling in the equations of the system. The other extreme is an uncoupled system, i.e. $m_{ij}\equiv0$ on Ω if $i\neq j$. Both, however, may be viewed as special cases of the following.

Condition 3.11. There is a finite sequence of row and column interchanges under which M is equivalent to a matrix of the form

$$\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_l \end{bmatrix} \tag{3.5}$$

where M_i is an $r_i\times r_i$ matrix, with $r_1+\cdots+r_l=r$. Furthermore, (2.1) is equivalent (under an appropriate relabelling of the u_i's) to the collection of systems

$$L_iv_i=\lambda M_iv_i,\ i=1,\cdots,l \tag{3.6}$$

where

$$v_i=\begin{bmatrix} w_1\\ \vdots \\ w_{r_i}\end{bmatrix}$$

has the property that $v_i\in\partial K_i$ and v_i a solution of (3.6) implies $v_i\equiv0$.

As is evident, not all systems of the form (1.1) satisfy Condition 3.11. However, there are sufficient conditions other than the above special cases under which Condition 3.11 holds. We will not dwell on these here.

Suppose now Condition 3.11 is satisfied. Let k_i, $i=1,\cdots,l$ be defined as follows: $k_1=0$, $k_i=\Sigma_{j<i}r_j$, $i=2,\cdots,l$. Then (1.3) can be equivalently expressed as

$$\begin{bmatrix} L_{k_i+1} & & \\ & \ddots & \\ & & L_{k_i+r_i} \end{bmatrix} \begin{bmatrix} u_{k_i+1} \\ \vdots \\ u_{k_i+r_i} \end{bmatrix} \tag{3.7}$$

$$= \begin{bmatrix} \lambda_{k_i+1} & & \\ & \ddots & \\ & & \lambda_{k_i+r_i} \end{bmatrix} \begin{bmatrix} m_{k_i+1,k_i+1} & \cdots & m_{k_i+1,k_i+r_i} \\ \vdots & & \vdots \\ m_{k_i+r_i,k_i+1} & \cdots & m_{k_i+r_i,k_i+r_i} \end{bmatrix} \begin{bmatrix} u_{k_i+1} \\ \vdots \\ u_{k_i+r_i} \end{bmatrix},$$

$i=1,\cdots,l$. Let F_i denote the function of Theorem 2.5 relative to (3.6) (i). We have the following result, which is similar in spirit to the results of [6].

THEOREM 3.12. *Suppose Condition* 3.11 *holds and that for each* $i=1,\cdots,l$, *there exists* $d(i)\in\{1,\cdots,r_i\}$ *such that* $(m_{k_i+d(i),k_i+d(i)})^+\neq 0$. *Assume also that* (ii) *of Theorem* 2.5 *holds for* $\{(\lambda_{k_{i+1}},\cdots,\lambda_{k_i+r_i})\in\mathbb{R}_+^{r_i}\colon |(\lambda_{k_{i+1}},\cdots,\lambda_{k_{i+r_i}})|=1\}$. *Then the set* $\{(\lambda_1,\cdots,\lambda_r)\in\mathbb{R}_+^r\colon$ (3.6) *has nontrivial solution in* $K\}=\cup_{i=1}^l T_i$, *where* $T_i=(\mathbb{R}_+)^{k_i}\times \operatorname{im}(F_i)\times(\mathbb{R}_+)^{r-(k_i+r_i)}$. *Furthermore, the geometric multiplicity of* $(\lambda_1,\cdots,\lambda_r)$ *is precisely the number of sets* T_i *of which* $(\lambda_1,\cdots,\lambda_r)$ *is a member.*

Proof. The result follows easily from Theorem 2.5, the results of §3, and the definition of Condition 3.11.

We conclude this section with a brief examination of the system

$$L_1u=\lambda(u+v), \qquad L_2v=\lambda v. \tag{3.8}$$

(3.8) is a typical example of a system for which Condition 3.11 fails to hold. Multiplicity results for more general upper-triangular nonsymmetric matrices M may be obtained in a manner analogous to that which follows. To this end, consider

$$L_1u=\lambda(u+v), \qquad L_2v=\mu v. \tag{3.9}$$

Let λ_1 and λ_2 represent the first eigenvalues for L_1 and L_2, respectively. If (3.8) has a nontrivial cone solution, either $v\equiv 0$ or $\mu=\lambda_2$. In the first case, $\lambda=\lambda_1$. If $\mu=\lambda_2$, however, it follows from the results of [12] that a nontrivial solution $\binom{u}{v}$ with $u>0$ and $v>0$ (note that $u=0$ implies $v=0$) is possible only in case $\lambda<\lambda_1$. In particular, it follows that if $\lambda_1=\lambda_2$, then $\lambda=\lambda_1=\lambda_2$ is a characteristic value of (3.8) which is geometrically but not algebraically simple.

4. Bifurcation results. In [11], Hess combined the result of Theorem 2.3 with the methods of [3] to obtain a bifurcation result for the nonlinear eigenvalue problem

$$Lu=\lambda A(u). \tag{4.1}$$

Here $A\colon K\to C_0(\overline{\Omega};\mathbb{R}^r)$ is the Nemytskii operator associated with a continuous function $a\colon \overline{\Omega}\times(\mathbb{R}^+)^r\to\mathbb{R}^r$. He assumed that a satisfies the following conditions:

$$a(x,0)=0, \qquad x\in\overline{\Omega}. \tag{4.2}$$

(4.3) There exists an $r\times r$ matrix m of functions $m_{kl}\in C(\overline{\Omega};\mathbb{R})$ such that

$$a(x,\sigma)=m(x)\sigma+o(|\sigma|)$$

as $|\sigma|\to 0$, $\sigma\in(\mathbb{R}^+)^r$ (uniformly for $x\in\overline{\Omega}$).

(4.4) There exists a number $\alpha \geqq 0$ such that

$$a(x,\sigma) \geqq -\alpha\sigma$$

for $(x,\sigma)\in\bar{\Omega}\times(\mathbb{R}^+)^r$.

Under the above conditions, Hess showed that if $\Sigma^+ = \{(\lambda,u)\in\mathbb{R}\times C_0^0(\bar{\Omega},\mathbb{R}^r)\colon \lambda>0,\ u\in K,\ Lu=\lambda A(u)\}$, Σ^+ contains an unbounded component Σ_0 emanating from $(\lambda^*,0)$, where $\lambda^*\geqq\bar{\lambda}$ ($\bar{\lambda}$ as in Theorem 2.3). He also identified certain cases when $\lambda^*=\bar{\lambda}$ (namely $m_{kl}\geqq 0$ for $k,\ l=1,\cdots,r$ or $m_{kl}\equiv 0$ if $k\neq l$).

Our results show that if $(M+\mu)(x_0)$ is irreducible for some $x_0\in\Omega$, then, at least locally, if $(\lambda,u)\in\Sigma_0$ and $u\neq 0$, then $u\in\operatorname{int}K$. Furthermore, if a is independent of x and $L_i=L_j$ for $i,j=1,\cdots,r$, Theorem 3.8 implies that also in this case $\lambda^*=\bar{\lambda}$.

We now establish some results on the multidimensionality of the nontrivial bifurcating solutions for nonlinear analogues of (1.3). The principal techniques for establishing such are the theorems of Alexander and Antman [1] and Fitzpatrick, Massabo, and Pejsachowitz [9]. Both results require that one work in an open subset of $\mathbb{R}^r\times E$, E an appropriate Banach space. (In this case, $[C_0^{1,\alpha}(\bar{\Omega},\mathbb{R})]^r$ is suitable, where $0<\alpha<1$, provided the coefficients of L_k and the m_{kl} are in $C^\alpha(\bar{\Omega},\mathbb{R})$. A precise definition of the spaces is found in [10].) A formulation based on [3] is not immediate. We therefore consider

$$Lu=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_r\end{bmatrix}Mu+H(\lambda_1,\cdots,\lambda_r,u), \tag{4.5}$$

where $H\colon R^r\times R^r\to R^r$ is continuous, cone-preserving and $H(\lambda_1,\cdots,\lambda_r,w_1,\cdots,w_r)=o(|(w_1,\cdots,w_r)|)$ (uniformly for $(\lambda_1,\cdots,\lambda_r)$ in compact subsets of R^r). It follows from [5] that the techniques of [1] are applicable to (4.5) provided $(\lambda_1,\cdots,\lambda_r)$ is an algebraically simple characteristic value of (1.3).

THEOREM 4.1. *Suppose there is* $k\in\{1,\cdots,r\}$ *such that* $m_{kk}^+\neq 0$ *and that for* $\mu\geqq 0$ *and sufficiently large* $(M+\mu)(x_0)$ *is nonnegative irreducible for some* $x_0\in\Omega$. (*If* $\mu\geqq 0$ *is required, assume that the conditions of Theorem* 3.8 *also hold*.) *Suppose that* $H(\lambda_1,\cdots,\lambda_r,-u)=-H(\lambda_1,\cdots,\lambda_r,u)$ *and that* F *is as in Theorem* 2.5. *Then there emanates from* $\operatorname{im}F\times\{0\}\cap(\mathbb{R}_+)^r\times E$ *a connected set* S *of solutions to* (4.5) *such that*

(i) $(\lambda_1,\cdots,\lambda_r,u)\in S$ *implies either* $u\in\operatorname{int}$ *or* $(\lambda_1,\cdots,\lambda_r,u)\in\operatorname{im}F\times\{0\}$;

(ii) $S\backslash(\operatorname{im}F\times\{0\})$ *is of topological dimension at least r at every point.*

Remark 4.2. (i) For a precise definition of topology dimension, see [1] and its references.

(ii) If $(\lambda_1,\cdots,\lambda_r)$ is restricted to a ray emanating from the origin of $\mathbb{R}^r$, the global bifurcation alternatives of Rabinowitz [15] hold. (See also [9].) It then follows that S is unbounded in the sense that $S\cap\partial((\mathbb{R}_+)^r\times C_0(\bar{\Omega},\mathbb{R}^r))\neq\varnothing$.

(iii) The oddness condition on H is a representative condition guaranteeing the existence of "small" positive solutions. Certainly, other such conditions are possible.

COROLLARY 4.3. *Suppose Condition* 3.11 *holds with* $l>1$. *Consider* (4.5) *and assume* H *of* (4.5) *has the form*

$$H(\lambda_1,\cdots,\lambda_r,u_1,\cdots,u_r)=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_r\end{bmatrix}\tilde{H}(u_1,\cdots,u_r).$$

Consider the problem

$$\begin{bmatrix} L_1 & & \\ & \ddots & \\ & & L_{r_1} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{r_1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{r_1} \end{bmatrix} M_1 \begin{bmatrix} u_1 \\ \vdots \\ u_{r_1} \end{bmatrix} + \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{r_1} \end{bmatrix} H^*(u_1,\cdots,u_{r_1}), \tag{4.6}$$

where

$$H^*(u_1,\cdots,u_{r_1}) = \begin{bmatrix} \tilde{H}_1(u_1,\cdots,u_{r_1},0,\cdots,0) \\ \vdots \\ \tilde{H}_{r_1}(u_1,\cdots,u_{r_1},0,\cdots,0) \end{bmatrix}.$$

Suppose that if $(\lambda_1,\cdots,\lambda_{r_1},u_1,\cdots,u_{r_1})$ *is a solution of* (4.6) *then* $(\lambda_1,\cdots,\lambda_{r_1},\lambda_{r_1+1},\cdots,\lambda_r,u_1,\cdots,u_{r_1},0,\cdots,0)$ *is a solution of* (4.5) *for any choice of* $\lambda_{r_1+1},\cdots,\lambda_r$. *Then if* M_1, H^* *and* F_1 *satisfy the hypotheses of Theorem* 4.1 *with respect to* (4.6), *there emanates from* $[\operatorname{im}F_1\times(\mathbb{R}^+)^{r-r_1}]\times\{0\}\subset(\mathbb{R}^+)^r\times[C_0^{1,\alpha}(\bar{\Omega},\mathbb{R})]^{r_1}\times\{0\}$ *a connected set of cone solutions to* (4.5) *such that* $(\lambda_1,\cdots,\lambda_r,u_1,\cdots,u_{r_1},0,\cdots,0)\in S$ *and* $(u_1,\cdots,u_{r_1})\neq 0$ *implies* $(u_1,\cdots,u_{r_1})\in\operatorname{int}K_1$. *Furthermore,* $S\setminus([\operatorname{im}F_1\times(\mathbb{R}^+)^{r-r_1}]\times\{0\})$ *has a topological dimension at least r at every point.*

Proof. If $(\lambda_{r_1+1},\cdots,\lambda_r)$ are considered fixed, Theorem 4.1 guarantees the existence of a set $\tilde{S}$ of solutions to (4.6) as above with topological dimension at least r_i. Then $S=\tilde{S}\times(\mathbb{R}^+)^{r-r_1}$.

Corollary 4.3 raises the obvious question: do there exist other "small" positive solutions with ($u_{r_1+1},\cdots,u_r$ not identically zero) bifurcating from $[\operatorname{im}F_1\times(\mathbb{R}^+)^{r-r_1}]\times\{0\}$? The answer is no, provided $(\lambda_1,\cdots,\lambda_r)$ is algebraically simple (with respect to all of M) and H is sufficiently well behaved. More precisely, we have the following result.

COROLLARY 4.4. *Suppose* $(\lambda_1,\cdots,\lambda_r)\in\operatorname{im}F_1\times(\mathbb{R}^+)^{r-r_1}$ *is as in Corollary* 4.3 *and that*

$$\dim N\left[I-\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}L^{-1}M\right] = \dim N\left[\left(I-\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}L^{-1}M\right)^2\right]=1.$$

Suppose also that there exists a continuous monotonic function G: $\mathbb{R}^+\to\mathbb{R}^+$ *such that* $G(0)=0$ *and for all* $v,w\in E=[C_0^{1,\alpha}(\bar{\Omega},\mathbb{R})]^r$

$$\|\tilde{H}(v_1,\cdots,v_r)-\tilde{H}(w_1,\cdots,w_r)\|_E \leq G\big(\|(v_1,\cdots,v_r)\|_E+\|(w_1,\cdots,w_r)\|_E\big)\|(v_1,\cdots,v_r)-(w_1,\cdots,w_r)\|_E.$$

Then near $(\lambda_1,\cdots,\lambda_r,0,\cdots,0)$ *in* $(\mathbb{R}^+)^r\times E$, *the solution set of* (4.5) *is as described by Corollary* 4.3.

Proof. The result follows from the generalization [1, Thm. 3.12] to several parameters of the global bifurcation theorem of Rabinowitz [16, Thm. 1.19].

We conclude this article by demonstrating the applicability of Corollaries 4.3 and 4.4 to the question of stable coexistence states in the Volterra–Lotka competition model with diffusion, recently studied by Cosner and Lazer [8], among others. The model in question is as follows:

$$(4.7)\qquad \begin{aligned} -\Delta u &= au - bu^2 - cuv, \\ -\Delta v &= dv - euv - fv^2, \end{aligned}$$

with $u(x)$ and $v(x)$ the population densities of the competing species at $x \in \Omega$, an open bounded smooth domain in $\mathbb{R}^n$, subject to $u \equiv 0 \equiv v$ on $\partial\Omega$. a and d, b and f and c and e are assumed to be positive constants, representing growth rates, self-regularization and competition, respectively, with positive diffusion coefficients normalized to 1. The only solutions to (4.7) with physical significance are those with $u \geqq 0$, $v \geqq 0$.

We will now consider (4.7) as a bifurcation problem with a and d acting as parameters and b, c, e, f considered fixed. Let λ_1 be the first eigenvalue of $-\Delta$ (relative to Ω and zero boundary conditions). Then if F_1 and F_2 are as in Theorem 3.8 relative to

$$(4.8)\qquad -\Delta u = au, \qquad -\Delta v = bv,$$

then $\operatorname{im}(F_1) \times R = \{(a,d) \colon a = \lambda_1\}$ and $R \times \operatorname{im}(F_2) = \{(a,d) \colon d = \lambda_1\}$.

Consider the problem

$$(4.9)\qquad \begin{aligned} -\Delta u &= au - bu^2 \quad &&\text{in } \Omega, \\ u &\equiv 0 &&\text{on } \partial\Omega. \end{aligned}$$

As noted in [8], (4.9) has a unique positive solution for all $a > \lambda_1$. In fact, one may realize these positive solutions as one of the two branches guaranteed by [16, Thm. 1.19]. To see this, note that the nonlinearity in the problem

$$(4.10)\qquad \begin{aligned} -\Delta u &= au - bu|u|, \quad &&\text{in } \Omega, \\ u &\equiv 0 &&\text{on } \partial\Omega, \end{aligned}$$

is odd. Since the eigenfunction corresponding to λ_1 is of one sign and since $f(x) = x|x|$ is continuously differentiable at $x = 0$, a positive branch for (4.10) is guaranteed at least locally. This branch coincides with solutions to (4.9). One may then use the uniqueness of the positive solutions, upper and lower solution techniques, the maximum principle, and global Rabinowitz bifurcation theory [15] to guarantee the continuation of the branch. Corollary 4.3 and Corollary 4.4 may now be applied. As a result, the only cone solution to (4.7) possible in a sufficiently small neighborhood of $(\lambda_1, d, 0, 0)$, $d \neq \lambda_1$, or of $(a, \lambda_1, 0, 0)$, $a \neq \lambda_1$, are of the form $(a, d, u, 0)$ or $(a, d, 0, v)$, respectively. Thus $a > \lambda_1$, $d > \lambda_1$ does not give a sufficient condition for stable coexistence states if $a \neq d$. This result strongly suggests stable coexistence states should be in general viewed as a secondary bifurcation phenomenon, as is the case in [4] and [7].

Note. It has been observed by one of the referees for this paper that the use of irreducibility in Lemma 3.1 is similar to that in the paper, G. J. Habeitler and M. A. Martino, *Existence theorems and spectral theory for the multigroup diffusion model*, Proc. Symposia in Applied Mathematics XI. Nuclear Reactor Theory, American Mathematical Society, Providence, RI, 1961, pp. 127–139.

REFERENCES

[1] J. C. ALEXANDER AND S. S. ANTMAN, *Global and local behavior of bifurcating multidimensional continua of solutions for multiparameter nonlinear eigenvalue problems*, Arch. Rat. Mech. Anal., 76 (1981), pp. 339–354.

[2] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), pp. 620–709.

[3] ______, *Nonlinear operators in ordered Banach space and some applications to nonlinear boundary value problems*, Lecture Notes in Mathematics 543, Springer, Berlin, 1976, pp. 1–55.

[4] J. BLAT AND K. J. BROWN, *Bifurcation of steady-state solutions in predator-prey and competition systems*, Proc. Roy. Soc. Edinburgh, 97A (1984), pp. 21–34.

[5] R. S. CANTRELL, *A homogeneity condition guaranteeing bifurcation in multiparameter nonlinear eigenvalue problems*, Nonlinear Anal., 8 (1984), pp. 159–169.

[6] ______, *Multiparameter bifurcation problems and topological degree*, J. Differential Equations, to appear.

[7] R. S. CANTRELL AND C. COSNER, *On the steady-state problem for the Volterra–Lotka competition model with diffusion*, Houston J. Math., to appear.

[8] C. COSNER AND A. C. LAZER, *Stable coexistence states in the Volterra–Lotka competition model with diffusion*, SIAM J. Appl. Math., 44 (1984), pp. 1112–1132.

[9] P. M.. FITZPATRICK, I. MASSABO AND J. PEJSACHOWITZ, *Global several-parameter bifurcation and continuation theorems: a unified approach via complementing maps*, preprint.

[10] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[11] P. HESS, *On the eigenvalue problem for weakly-coupled elliptic systems*, Arch. Rat. Mech. Anal., 81 (1983), pp. 151–159.

[12] P. HESS AND T. KATO, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differential Equations, 5 (1980), pp. 999–1030.

[13] R. NUSSBAUM, *Periodic solutions of some nonlinear integral equations*, Proc. Int. Symp. on Dynamical Systems, A. R. Bednarek and L. Cesari, eds., Gainesville, FL, 1976.

[14] M. H. PROTTER AND H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[15] P. H. RABINOWITZ, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math., 3 (1973), pp. 161–201.

[16] ______, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., 77 (1971), pp. 487–513.

[17] F. RELLICH, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, New York, 1969.

[18] K. SCHMITT AND H. L. SMITH, *Positive solutions and conjugate points for systems of differential equations*, Nonlinear Anal., 2 (1978), pp. 93–105.

SIAM J. MATH. ANAL.
Vol. 17, No. 4, July 1986

007

ON THE EIGENVALUE PROBLEM FOR COUPLED ELLIPTIC SYSTEMS*

ROBERT STEPHEN CANTRELL[†] AND KLAUS SCHMITT[‡]